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# An extended Kays and Crawford turbulent Prandtl number model

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**Abstract**—For theoretical predictions of turbulent heat transfer in boundary layers and in duct flows, the knowledge of the turbulent Prandtl number is crucial. This is especially true for fluids with low molecular Prandtl numbers (liquid metals). The present formulation, which is an extended Kays and Crawford (*Convective Heat and Mass Transfer*, 3rd edn. McGraw-Hill, New York, 1993) turbulent Prandtl number model, can be used for accurately predicting the heat transfer for liquid metal flows. The Nusselt numbers calculated with the modified model for $Pr_t$ are found to be in good agreement with experimental data for fully-developed pipe flows as well as for thermally developing pipe flows for various wall boundary conditions. The present model reduces to the one given in the above reference for liquids and gases with higher molecular Prandtl numbers. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

The knowledge of the turbulent Prandtl number is a central problem of all theoretical considerations concerning the turbulent heat transfer in two-dimensional boundary layers or in duct flows. A large number of models have been published in the past which address the prediction of the turbulent Prandtl number for such situations. Reviews of the existing work can be found in Reynolds [1] and in Kays [2]. In general it can be noted that:

$$Pr_t \leqslant 1 \quad \text{for } Pr \geqslant 1 \quad \text{(gases and liquids)}$$

$$Pr_t \gg 1 \quad \text{for } Pr \ll 1 \quad \text{(liquid metals)}.$$

There is a very strong influence of the molecular Prandtl number on the value of $Pr_t$ for fluids with very low Prandtl numbers (liquid metals). Additionally, there is an influence of the wall distance on the value of the turbulent Prandtl number which tends to increase $Pr_t$ close to the wall. This increase in $Pr_t$ near the wall is especially important for high Prandtl number fluids because of the very thin thermal boundary layer. Outside this thin layer near the wall the turbulent Prandtl number seems to be constant for $Pr > 1$. Because of the strong dependence of $Pr_t$ on molecular Prandtl number for low Prandtl number fluids (liquid metals) effort has been taken in the past to develop prediction models for $Pr_t$ which are able to take the dependence of $Pr_t$ on the molecular Prandtl number as well as the dependence of $Pr_t$ on the wall distance into account. Based on a purely empirical basis, Notter [3] developed a model for $Pr_t$ for low Prandtl number flows which was used for the recalculation of his measured Nusselt numbers. Jenkins [4] suggested that a turbulent eddy, while moving transverse to the mean direction of flow, might lose heat at a different rate than it loses momentum. Further reflections on the idea of Jenkins [4] resulted in a category of models for $Pr_t$ which introduced a discrete element which gains or loses heat and momentum as it moves through the fluid. The model of Azer and Chao [5] falls into this category. This model is often used to prescribe $Pr_t$ for liquid metal flows. Additionally Kays and Crawford [6] developed a prediction model for $Pr_t$ which can be used for all molecular Prandtl numbers [2]. The model contains two empirical constants which must be determined from available experimental data. Cebeci [7] adopted van Driest's idea of near-wall damping of the mixing length to propose a turbulent Prandtl number concept. This model was extended by Chen and Chiou [8] for liquid metal flow in pipes by incorporating the enthalpy thickness $\delta_T$ in $Pr_t$. Although the predictions of Chen

## NOMENCLATURE

| | | | |
|---|---|---|---|
| $a$ | thermal diffusivity | $T$ | temperature |
| $D$ | pipe diameter | $T_W$ | wall temperature |
| $l$ | mixing length | $T_b$ | bulk temperature |
| $Nu$ | Nusselt number | $u$ | axial velocity |
| | $= -(\partial T/\partial r)_{r=R} D/(T_W - T_b)$ | $\bar{u}_0$ | axial mean velocity |
| $Nu_\infty$ | Nusselt number for fully-developed flow | $u_\tau$ | friction velocity $= \sqrt{(\tau_{rx})_W/\rho}$ |
| $p$ | pressure | $y^+$ | wall coordinate $= u_\tau(R-r)/\nu$ |
| $Pr$ | Prandtl number | $x$ | axial coordinate. |
| $Pe$ | Peclet number $= Re\,Pr$ | | |
| $Pe_t$ | turbulent Peclet number, equation (9) | Greek symbols | |
| $Pr_t$ | turbulent Prandtl number | $\varepsilon_m$ | eddy kinematic viscosity |
| $Pr_{t\infty}$ | turbulent Prandtl number at infinity | $\rho$ | density |
| $r$ | radial coordinate | $\lambda_n$ | eigenvalue |
| $Re$ | Reynolds number $= \bar{u}_0 D/\nu$ | $\nu$ | kinematic viscosity. |

and Chiou agree well with measured Nusselt numbers for liquid metal flows, their model has the disadvantage that an additional nonlinearity is introduced into the energy equation by using the enthalpy thickness $\delta_T$ in $Pr_t$. This restricts the possibility of analytical solutions of the energy equation for hydrodynamically fully-developed flows. Furthermore, the von Karman constant is changed in the mixing length expression for their calculation. This might lead to misleading results for larger molecular Prandtl numbers. Jischa and Rieke [9] developed a model for $Pr_t$ from the modeled transport equations for the turbulent kinetic energy and for the turbulent heat flux. Their final result for the turbulent Prandtl number for pipe and channel flows is given by:

$$Pr_t = 0.9 + \frac{182.4}{Pr\,Re^{0.888}}. \tag{1}$$

Although the prediction model of Kays and Crawford [6] has been developed also for liquid metal flows, it seems to overpredict Nusselt numbers for this case, whereas it predicts Nusselt numbers for air flow ($Pr \approx O(1)$) which are in excellent agreement with experimental data [6]. The purpose of the present paper is to modify the turbulent Prandtl number model of Kays and Crawford [6] to predict more accurately the heat transfer for low Prandtl number fluids in pipe and channel flows. By doing so, a model for $Pr_t$ will be obtained which can be used for all molecular Prandtl numbers. In the present study, heat transfer in pipe flow is investigated for fully-developed flow conditions and also for thermal developing flow for different wall boundary conditions.

## 2. ANALYSIS

Figure 1 shows the geometrical configuration and the coordinate system. The turbulent flow enters the pipe at $x = 0$ with a fully developed velocity profile and constant temperature $T_0$. By assuming an incompressible Newtonian fluid with constant fluid properties, the steady-state conservation equations can be written as:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(1+\frac{\varepsilon_m}{\nu}\right)\frac{\mathrm{d}u}{\mathrm{d}r}\right] \tag{2}$$

$$u\frac{\partial T}{\partial x} = \frac{a}{r}\frac{\partial}{\partial r}\left[r\left(1+\frac{Pr\,\varepsilon_m}{Pr_t\,\nu}\right)\frac{\partial T}{\partial r}\right]. \tag{3}$$

The boundary conditions belonging to equations (2) and (3) are given by:

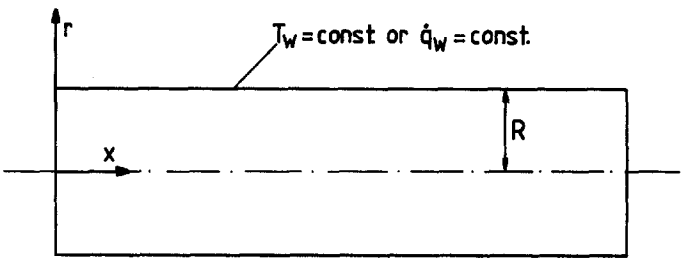


Fig. 1. Geometrical configuration and coordinate system.

$$x = 0: \quad u = f(r) \qquad T = T_0$$
$$r = 0: \quad du/dr = 0 \qquad \partial T/\partial r = 0$$
$$r = R: \quad u = 0$$
$$T = T_w \quad \text{(const. wall temperature)} \quad \text{or}$$
$$\dot{q}_w = -k\partial T/\partial r \quad \text{(const. wall heat flux)}. \tag{4}$$

Additionally the conservation of mass in integral form

$$2\bar{u}_0 R = \int_0^R ur\,dr \tag{5}$$

has to be satisfied. This equation determines the pressure drop in the pipe. The eddy viscosity $\varepsilon_m$ in equation (2) is modeled by using the well-known Nikuradse mixing length formula with the van Driest damping factor:

$$\varepsilon_m = l^2\left|\frac{\partial u}{\partial r}\right| \tag{6}$$

with $l$ given by

$$l = R\left(0.14 - 0.08\left(\frac{r}{R}\right)^2 - 0.06\left(\frac{r}{R}\right)^4\right) \times (1 - \exp(-y^+/26)). \tag{7}$$

2.1. *Model for the turbulent Prandtl number*

In order to solve the above given energy equation (3), a model for $Pr_t$ has to be specified. The present results are based on the turbulent Prandtl number model of Kays and Crawford [6], which is given by:

$$Pr_t = \frac{1}{\frac{1}{2Pr_{t\infty}} + C\,Pe_t\sqrt{\frac{1}{Pr_{t\infty}}} - (C\,Pe_t)^2\left[1 - \exp\left(-\frac{1}{C\,Pe_t\sqrt{Pr_{t\infty}}}\right)\right]} \tag{8}$$

where

$$Pe_t = Pr\frac{\varepsilon_m}{\nu}. \tag{9}$$

$Pr_{t\infty}$ is the value of $Pr_t$ far away from the wall and $C = 0.3$ is a constant prescribing the spatial distribution of $Pr_t$ vs $Pe_t$. If $Pe_t \to \infty$, $Pr_t$ according to equation (8) tends to $Pr_{t\infty}$, which can be shown by expanding the denominator of equation (8) for large $Pe_t$. This will normally be the case for large values of $y^+$. At the wall, $Pe_t = 0$ which implies that $Pr_t \to 2Pr_{t\infty}$ as can be seen by examining the denominator of equation (8) for small $Pe_t$. In the model originally proposed by Kays and Crawford [6] $Pr_{t\infty}$ is fixed to a constant value of 0.85. This has the disadvantage that $Pr_t$ is always smaller than 1.7 and introduces an undesired behavior into the model, because for $Pr \to 0$ heat will be transferred exclusively by molecular conduction and, therefore, $Pr_t$ should tend to a large value. As mentioned before, Jischa and Rieke [9] developed a model for $Pr_t$ from modeled transport equations. They obtained the relationship for $Pr_t$ which is given by equation (1). This equation does not consider the spatial distribution of $Pr_t$. Therefore, the value for $Pr_t$ given by equation (1) can be seen as a mean value of $Pr_t$ across the whole boundary layer. One possible way to overcome the above-mentioned undesired behavior in the model of Kays and Crawford [6] is to use the functional form of $Pr_t$ developed by Jischa and Rieke [9] as an approximation for $Pr_{t\infty}$ in equation (8). Of course, both constants in equation (1) will need some readjustment, since these constants have been evaluated by Jischa and Rieke [9] from experimental data. Nevertheless, the functional relationship between $Pr_{t\infty}$ and $Re$ and $Pr$ given by equation (1) will stay the same. Therefore, $Pr_{t\infty}$ is modeled as:

$$Pr_{t\infty} = 0.85 + \frac{D}{Pr\,Re^{0.888}} \tag{10}$$

with the constant $D$, to be determined by one single experiment. The second constant in equation (10) was set to 0.85, because there is evidence from experimental data that $Pr_{t\infty}$ will attain this value for high $Re$ and $Pr$ [2]. Rieke and Jischa [9] determined the constant $D = 182.4$ in equation (1) from experimental data of Fuchs [10] for liquid Na. Because of the very low Prandtl number fluid, which was used for the experiments of Fuchs [10] ($Pr = 0.007$), there is only a small spatial variation in $Pr_t$ across the pipe diameter. Therefore, the constant $D$ in equation (10) can be easily determined by comparing both models for $Pr_t$ for one single experiment of Fuchs [10] ($Re = 141\,000$). By doing so, a value for $D$ of 100 can be obtained. Therefore, $D = 100$ was used for all following calculations, even though further adjusting of the constant $D$ might even lead to better agreement between calculations and measurements.

It should be noted here, that the above given model for $Pr_t$, equations (8)–(10) will only give different values for $Pr_t$ in comparison to the original Kays and Crawford model for low Prandtl number flows. For liquids and gases as working fluids, the above given expressions reduce to the original model of Kays and Crawford [6].

## 3. RESULTS AND DISCUSSION

3.1. *Fully developed flow*

3.1.1. *Constant wall heat flux.* For the case of constant heat flux at the pipe wall two sources of reliable data are the one of Skupinski *et al.* [11] for liquid NaK and the one of Fuchs [10] for liquid Na. Figure 2 shows a comparison between the experimental data of Skupinski *et al.* [11] and calculations using the modified model for $Pr_t$. It can be seen that the agree-

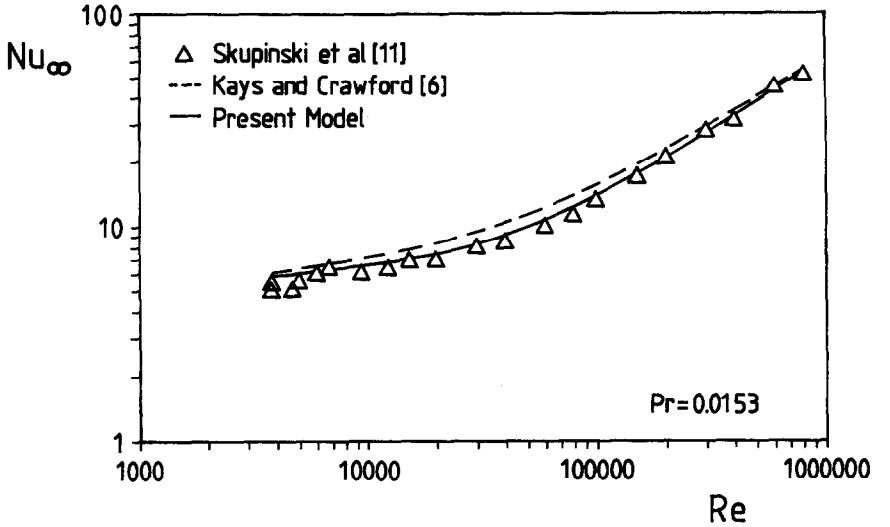


Fig. 2. Nusselt number for fully-developed pipe flow ($\dot{q}_w$ = const.) as a function of the Reynolds number.

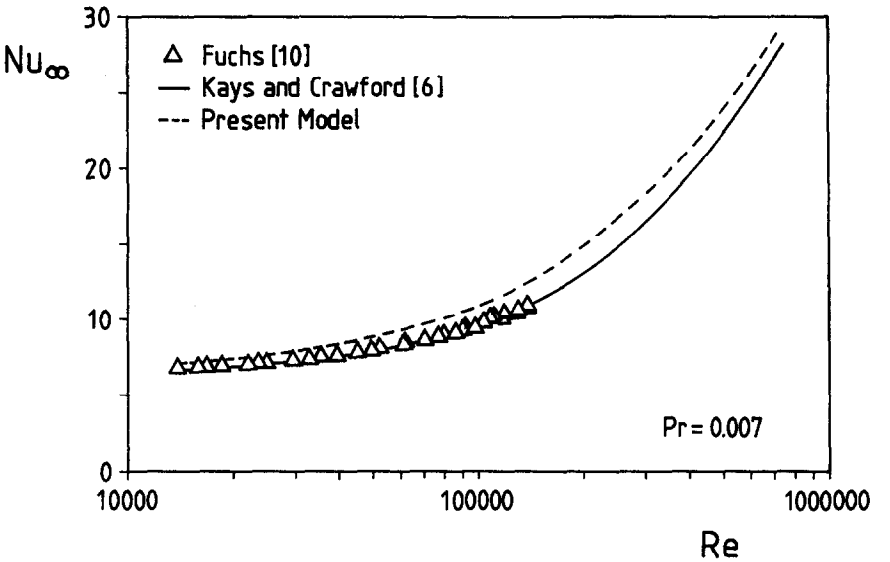


Fig. 3. Nusselt number for fully-developed pipe flow ($\dot{q}_w$ = const.) as a function of the Reynolds number.

ment between calculations and experimental data is very good. Nusselt numbers predicted with the original model for $Pr_t$ are plotted for comparison in Fig. 2. It can be seen that the original model gives always slightly higher values for $Nu_\infty$ than the model proposed here. Figure 3 shows a comparison between experimental data of Fuchs [10] and calculations. As can be seen from Fig. 3, the agreement between predicted Nusselt numbers using the modified model for $Pr_t$ and experimental data is excellent. Additionally it is noted that the original model for $Pr_t$ overpredicts the fully developed Nusselt number.

3.1.2. *Constant wall temperature.* According to Reed [12] only the experimental data of Sleicher *et al.* [13] and Gilliland *et al.* [14] (taken from Azer and Chao [5]) are judged to be reliable for the case of uniform wall temperature for liquid metal flows. Figure 4 shows a comparison between predicted Nusselt numbers for fully-developed pipe flow and experimental data of [13] and [14]. It can be seen that the original model for $Pr_t$ of Kays and Crawford [6] results in somewhat too high values for $Nu_\infty$, whereas the model with the changed formulation for $Pr_{t\infty}$ predicts Nusselt numbers which are in closer agreement with the measured values. For comparison, Nusselt numbers, calculated with the turbulent Prandtl number model of Azer and Chao [5] are plotted additionally in Fig. 4. It can be seen that the Nusselt numbers predicted with this model are a little bit too low for high values of the Peclet number. Interestingly, nearly

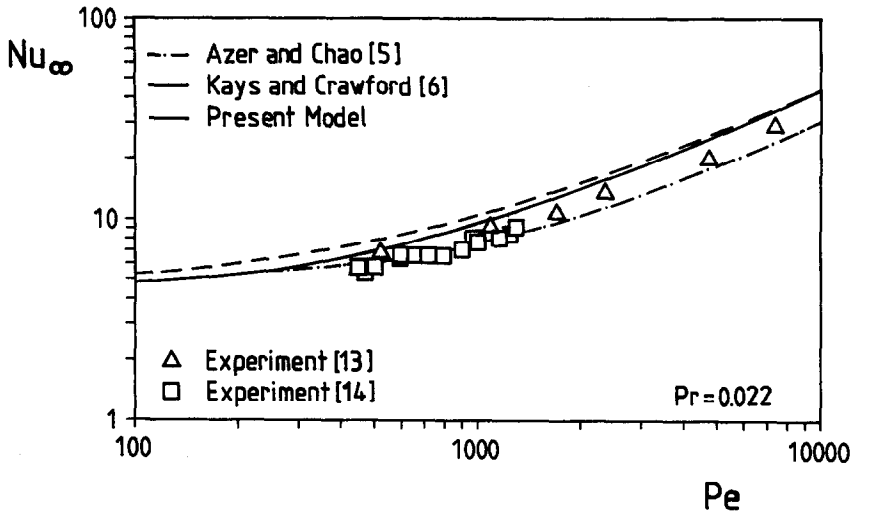


Fig. 4. Nusselt number for fully-developed pipe flow ($T_w$ = const.) as a function of the Peclet number.

the same values for $Nu_\infty$ are obtained by both models for values of the Peclet number below 500.

3.2. *Thermally developing flow with a hydrodynamically fully developed velocity profile*

For flow of liquid metals, the thermal entry length tends to be larger, similar to laminar pipe flow. Therefore, it is important to see if the proposed model for $Pr_t$ is also able to predict the heat transfer characteristics correctly for thermally developing flow. For this kind of problem the energy equation (3) can be solved analytically (turbulent Graetz problem) and the distribution of the local Nusselt number can be expressed as infinite sums. The final expressions are given by:

- Constant wall heat flux

$$Nu = \frac{-2}{A + \sum_{n=0}^{\infty} B_n \exp(-\lambda_n^2 \tilde{x})}. \tag{11}$$

- Constant wall temperature

$$Nu = \frac{\sum_{n=0}^{\infty} C_n \exp(-\lambda_n^2 \tilde{x})}{\sum_{n=0}^{\infty} \frac{C_n}{\lambda_n^2} \exp(-\lambda_n^2 \tilde{x})}. \tag{12}$$

The constants $B_n$ and $C_n$ as well as the eigenvalues $\lambda_n$ are obtained from the solution of the corresponding eigenvalue problem. For more details concerning the solution procedure the reader is referred to Kays and Crawford [6].

3.2.1. *Constant wall heat flux.* Figure 5 shows a comparison of experimental data taken from the paper of Chiou and Chen [8] with a calculation for constant wall heat flux boundary condition. From Fig. 5 it can be seen that the calculation agrees well with the experimental data.

3.2.2. *Constant wall temperature.* For the case of constant wall temperature, Awad [15] reported data for the local Nusselt number in the thermal developing

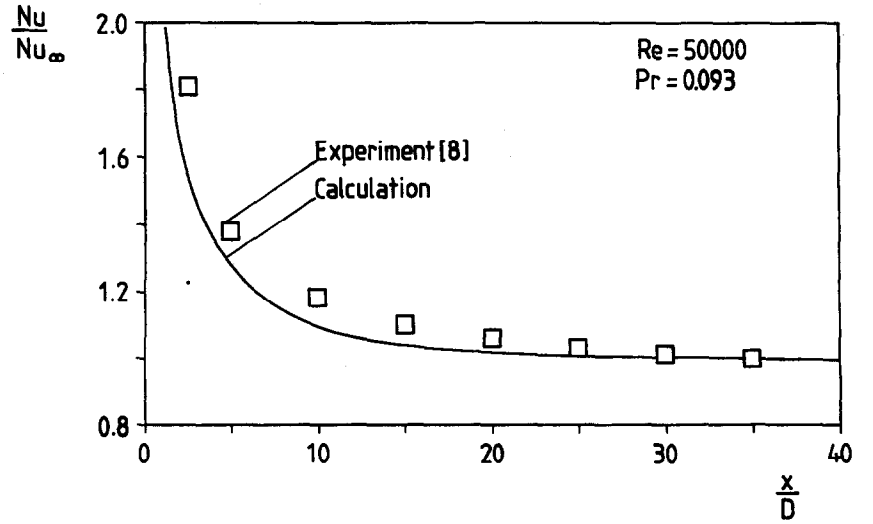


Fig. 5. Variation of the local Nusselt number in the thermal entry region of a circular pipe ($\dot{q}_w$ = const.).

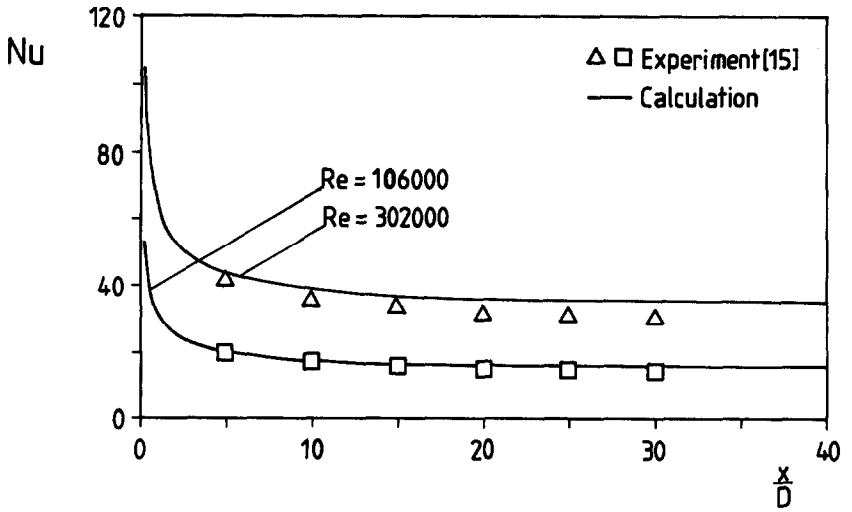


Fig. 6. Variation of the local Nusselt number in the thermal entry region of a circular pipe ($T_w$ = const., $Pr = 0.022$).

region. Figure 6 shows comparisons between predicted Nusselt numbers and experimental data of Awad [15]. From Fig. 6 it is evident that the calculations are in good agreement with the measured values for both shown Reynolds numbers.

## 4. CONCLUSIONS

According to the present study, the following major conclusions can be drawn:

- The original model for $Pr_t$ of Kays and Crawford [6] uses a constant value for $Pr_{t\infty} = 0.85$. This might lead to the prediction of an undesired behaviour in $Pr_t$ for $Pr \to 0$, because in this case $Pr_t$ should tend to a large value, indicating that only molecular heat conduction takes place in the flow. By using the proposed function for $Pr_{t\infty}$ in the original model of Kays and Crawford this shortcoming can be eliminated.
- The Nusselt numbers predicted with the modified model for $Pr_t$ are in good agreement with experimental data for fully-developed flow as well as for thermally developing flow.
- The proposed modification does not lead to additional nonlinearities in the energy equation as is the case for the model of Chiou and Chen [8]. Therefore, existing analytical solutions can be used for the resulting turbulent Graetz problem.

Finally, it should be noted that the model for $Pr_t$ of Kays and Crawford [6] can be used for all types of molecular Prandtl numbers. The reader is referred to Kays [2] who explained the usage of the original model of Kays and Crawford for higher values of the molecular Prandtl number.

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